Rotation Minimizing vector fields and frames in Riemannian manifolds

Fernando Etayo*

Abstract

We prove that a normal vector field along a curve in $\mathbb{R}^3$ is rotation minimizing (RM) if and only if it is parallel respect to the normal connection. This allows us to generalize all the results of RM vectors and frames to curves immersed in Riemannian manifolds.

AMS Classification: 57R25, 53C40, 53A04.
Keywords: Rotation minimizing, Riemannian manifold, Normal curvature.

1 Introduction

In a celebrated paper [1], Bishop introduced what nowadays are called rotation minimizing vector fields (RM, for short) over a curve in the Euclidean 3-space. The purpose of this note is to show how they can be defined in Riemannian manifolds.

When one considers vector fields and moving frames along a curve, one can take into account two general ideas: defining an adapted frame whose first vector is the tangent vector to the curve and defining a frame whose first vector is the position vector of the point of the curve. In both cases, one can consider Frenet frames and RM frames. See [4]. The first idea, frames containing the tangent vector of the curve as first vector, can be generalized to the case of a curve immersed in a Riemannian manifold, while the second one has no sense in this general framework.

We shall introduce such definition of a RM vector field over a curve in a Riemannian manifold as a vector field parallel respect to the normal connection. It will be shown that this definition is consistent with that of Bishop for curves in the Euclidean space (section 4), and that remains invariable under isometries (section 5). In section 2 we shall remember the basic definitions about RM vector fields and frames, and in section 3 about curves in Riemannian manifolds.

2 RM vector fields and frames of a curve in $\mathbb{R}^3$

Definition 1 *A normal vector field $\vec{v} = \vec{v}(t)$ over a curve $\gamma = \gamma(t)$ in $\mathbb{R}^3$ is said to be* relatively parallel *or* rotation minimizing *(RM) if the derivative $\vec{v}'(t)$ is proportional to $\gamma'(t)$.*

*Departamento de Matemáticas, Estadística y Computación. Facultad de Ciencias. Universidad de Cantabria. Avda. de los Castros, s/n, 39071 Santander, SPAIN. e-mail: etayof@unican.es

Remark 2 *(1) In this case the ruled surface* $f(t,\lambda) = \gamma(t) + \lambda\vec{v}(t)$ *is developable, because* $[\gamma'(t), v(t), v'(t)] = 0$.

(2) If v *is a RM vector field, then* $\parallel \vec{v} \parallel$ *is constant. Let* $\vec{t}$ *denote the tangent vector to* γ*. Then* $\vec{v}' = \lambda\vec{t} \Rightarrow \vec{v}' \perp \vec{v} \Rightarrow \frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$.

Example 3 *(1) Let* $\gamma(t) = t$ *be the line given by the x-axis in* $\mathbb{R}^3$*. Then a normal vector field* $\vec{v}(t)$ *over* γ *is RM respect to* γ *iff it is constant.*

(2) Let $\gamma(t) = (\cos t, \sin t, 0)$ *the unit circle in the horizontal plane. Let* $(0,0,h)$ *any point in the vertical axis. Let us consider the vector* $\vec{v}(t)$ *joining* $x(t)$ *and* $(0,0,h)$*. Then* $\vec{v}$ *is RM. The developable surface generated is the corresponding cone.*

(3) Normal and binormal vector fields, $\vec{n}$ *and* $\vec{b}$*, of a Frenet moving frame are not RM vector fields in general. Let* $\gamma = \gamma(s)$ *a curve parametrized respect to the arc-length. Then, the Frenet-Serret formulas say that* $\vec{n}' = -\kappa\vec{t} + \tau\vec{b}$*, and* $\vec{b} = -\tau\vec{n}$ *where* κ *and* τ *denote the curvature and the torsion. For a twisted (non plane) curve,* $\tau \neq 0$*, the above equations show that* $\vec{n}$ *and* $\vec{b}$ *are nor RM vector fields. This is the case, for instance, of the helix* $(a\cos t, a\sin t, bt)$*. The normal vector is* $(-\sin t, \cos t, 0)$ *and the surface generated by the normal lines is the helicoid, which is not developable, because its Gauss curvature does not vanish. For a plane curve,* $\vec{n}$ *and* $\vec{b}$ *are RM vector fields.*

Definition 4 *Let* $\gamma = \gamma(t)$ *in* $\mathbb{R}^3$ *be a curve. A RM* frame, parallel frame, natural frame, Bishop frame *or* adapted frame *is a moving orthonormal frame* $\{\vec{t}(t), \vec{u}(t), \vec{v}(t)\}$ *along* γ*, where* $\vec{t}(t)$ *is the tangent vector to* γ *at the point* $\gamma(t)$ *and* $\vec{u}, \vec{v}$ *are RM vector fields.*

Example 5 *If* $\vec{u}$ *is a unitary RM vector field along* γ*, then* $\{\vec{t}, \vec{u}, \vec{t}\times\vec{u}\}$*is a RM frame along* γ*.*

RM frames are a very useful tool in some aspects of Computer Aided Geometric Design. See [3] as a basic reference.

3 Curves immersed in a Riemmanian manifold

In this section we shall remember the main facts about curves immersed in a Riemmanian manifold.

As it is well known, if M is a submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$, then the Levi-Civita connection $\overline{\nabla}$ of $(\overline{M}, \overline{g})$ induces a Levi-Civita connection in $(M, g = \overline{g}|_M)$ and a normal connection $D^\perp : \mathfrak{X}(M) \times \Gamma TM^\perp \to \Gamma TM^\perp$, where $\mathfrak{X}(M)$ denotes the module of vector fields of the submanifold and $\Gamma TM^\perp$ the module of sections of the normal bundle. We shall use the notation of [5, § VIII].

The normal connection is defined as follows. First of all, consider a point $p \in M$. Then the tangent space at the manifold $\overline{M}$ can be decomposed as a orthogonal direct sum $T_p(\overline{M}) = T_pM \oplus T_p^\perp M$, where T_pM (resp. $T_p^\perp M$) denotes the tangent (resp. the normal) space to the submanifold. The set $TM = \bigcup_{p\in M} T_pM$ (resp. $T^\perp M = \bigcup_{p\in M} T_p^\perp M$) is a manifold called the tangent bundle (res. the normal bundle) and it has a a structure of vector bundle over M, given

by the natural projection $\pi : TM \to M; \pi(u) = p$ if $u \in T_pM$ (resp. $\pi^\perp : TM^\perp \to M; \pi(v) = p$ if $v \in T_p^\perp M$). Let us denote by $\mathfrak{X}(M)$ (resp. $\Gamma TM^\perp$) the module of vector fields over M, i.e., the module of sections of $\pi : TM \to M$ (resp. the module of sections of $\pi^\perp : TM^\perp \to M$).

Then for any $X \in \mathfrak{X}(M)$ and $v \in \Gamma TM^\perp$ one has the decomposition

$$\overline{\nabla}_X v = -A_v X + D_X^\perp v$$

where $-A_v X \in \mathfrak{X}(M)$ and $D_X^\perp v \in \Gamma TM^\perp$. The *Weingarten operator* A is a $\mathfrak{F}(M)$-bilinear map and the normal connection $D^\perp$ is a connection in the normal bundle $T^\perp M \to M$.

Moreover, if $v, w \in \Gamma TM^\perp$ are two normal vector fields, then

$$\overline{g}(D_X^\perp v, w) + \overline{g}(v, D_X^\perp w) = X(\overline{g}(v, w))$$

which shows that the normal connection $D^\perp$ is metric for the fibre metric in the normal bundle $TM^\perp$.

A normal vector field v is said to be *parallel* respect to $X \in \mathfrak{X}(M)$ if $D_X^\perp v = 0$.

Let $\pi : E \to M$ be a vector bundle with a connection D, compatible with a metric on E. The notion of being a section $\sigma : M \to E$ parallel respect to the connection D is equivalent to that it is defined by the parallel transport induced by the connection. See [9] for details. The parallel transport defines an isometry between any two different fibres of $\pi : E \to M$. Thus, the norm of a parallel section remains constant and the angle between two parallel sections also remains constant. In the present case of having a submanifold M of a Riemannian manifold $(\overline{M}, \overline{g})$, the vector bundle is the normal bundle, and the metric is the restriction of $\overline{g}$ to normal vectors.

Many results about curves un Riemannian manifolds have been obtained in the past. We would like to to point out that generalizations of Frenet formula have been obtained in [6], in [7] for the case of spaces of constant curvature, and in [8] for the case of the Minkowski space. Besides in [2] some results about the total curvature of a curve in a Riemannian manifold are also obtained.

4 RM vector fields over a curve immersed in a Riemmanian manifold

The notion of RM vector field implies a notion of parallel transport. We shall show this carefully.

Let us consider the case where M is a curve γ and $(M, g = \overline{g}|_M)$ is $\mathbb{R}^3$ with the standard product. Let us denote by $T_{\gamma(t_0)}$ (resp. $T_{\gamma(t_0)}^\perp$) the tangent line (resp. the normal plane) to γ in $\gamma(t_0)$. Then we have:

Theorem 6 *A normal vector field v over a curve γ immersed in $\mathbb{R}^3$ is a RM vector field iff ir is parallel respect to the normal connection of γ.*

Proof. Let us denote as (x^1, x^2, x^3) the global coordinates in $\mathbb{R}^3$. The curve γ can be expressed as $\gamma(s) = (\gamma^1(s), \gamma^2(s), \gamma^3(s))$, s being the arc-length parameter, and the tangent vector is $\overline{t} = \gamma'(s) = \frac{\partial}{\partial x^i}\frac{d\gamma^i}{ds}$ (Einstein's convention is assumed).

Let v be a normal vector field over γ, $v = \frac{\partial}{\partial x^i} v^i$. The condition of being normal to the curve means that

$$\sum_{i=1}^{3} v^i \ \frac{d\gamma^i}{ds} = 0$$

The condition of being v a RM vector field means that $v'(t)$ is proportional to $\gamma'(t)$, i.e.

$$\frac{dv^i}{ds} = \lambda(s) \ \frac{d\gamma^i}{ds}, \qquad \forall i = 1, 2, 3$$

where $\lambda = \lambda(s)$ is a function.

Now, we shall check the value of $D^{\perp}_{\gamma'(s)} v$. We must prove that $D^{\perp}_{\gamma'(s)} v = 0$ iff the above equation is satisfied. Let $\overline{\nabla}$ be the Levi-Civita connection of $\mathbb{R}^3$. The all of its Christoffel symbols vanish, and then one has:

$$\overline{\nabla}_{\overline{t}} \ v = \overline{\nabla}_{\frac{\partial}{\partial x^i}\frac{d\gamma^i}{ds}} \left(\frac{\partial}{\partial x^j} v^j\right) = \frac{d\gamma^i}{ds}\overline{\nabla}_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} v^j\right) = \frac{\partial}{\partial x^j}\frac{d\gamma^i}{ds}\frac{\partial v^j}{\partial x^i}$$

Applying the chain rule one has:

$$\frac{\partial v^j}{\partial x^i} = \frac{dv^j}{ds}\frac{ds}{dx^i} = \frac{dv^j}{ds}\left(\frac{dx^i}{ds}\right)^{-1} = \frac{dv^j}{ds}\left(\frac{d(x^i \circ \gamma^{-1})}{ds}\right)^{-1} = \frac{dv^j}{ds}\left(\frac{d\gamma^i}{ds}\right)^{-1}.$$

And then,

$$\overline{\nabla}_{\overline{t}} \ v = \frac{\partial}{\partial x^j}\frac{d\gamma^i}{ds}\frac{\partial v^j}{\partial x^i} = \frac{\partial}{\partial x^j}\frac{d\gamma^i}{ds}\frac{dv^j}{ds}\left(\frac{d\gamma^i}{ds}\right)^{-1} = \frac{\partial}{\partial x^j}\frac{dv^j}{ds}.$$

Finally, v is parallel $\Leftrightarrow D^{\perp}_{\gamma'(s)} v = 0 \Leftrightarrow \overline{\nabla}_{\overline{t}} v$ is tangent to $\gamma \Leftrightarrow \frac{dv^i}{ds} = \lambda(s) \ \frac{d\gamma^i}{ds}, \forall i = 1, 2, 3 \Leftrightarrow v$ is a RM vector field, thus finishing the proof. □

The above result is important because it allows to obtain the definition of a RM vector field over a curve immersed in a Riemmannian manifold. Moreover, one easily can deduce the following properties of the above Proposition.

Corollary 7 *With the above notation:*

(1) Given a vector $v_0 \in T^{\perp}_{\gamma(t_0)}$ *there exists a unique RM vector field* v *over* γ *such that* $v(t_0) = v_0$.

(2) If v *is a RM vector fields over* γ *then the norm* $\|v\|$ *is constant.*

(3) If v *and* w *are RM vector fields over* γ *then the angle between* $v(t)$ *and* $w(t)$ *is constant.*

□

Thus, we can give the following

Definition 8 *Let γ be a curve immersed in a Riemannian manifold $(\overline{M}, \overline{g})$.*

(1) A normal vector field v over γ is said to be a RM vector field *if it is parallel respect to the normal connection of γ.*

(2) A parallel frame*,* natural frame*, RM* frame *or* adapted frame *is a moving orthonormal frame $\{\vec{t}(t), \vec{v}_1(t), \ldots, \vec{v}_n(t)\}$ along γ, where $\vec{t}(t)$ is the tangent vector to γ at the point $\gamma(t)$ and $\vec{v}_i$ are RM vector fields, $\forall i \in 1, \ldots, n$.*

If one defines an orthonormal frame $\{\vec{t}(t_0), \vec{v}_1(t_0), \ldots, \vec{v}_n(t_0)\}$ at a point $\gamma(t_0)$ of a curve γ then by parallel transport it can be extended along γ. Parallel transport is an isometry, this meaning that norms and angles are preserved.

5 RM frames and transformations

First of all, we prove that RM vector fields and frames are preserved by isometries. Let $\mu : (\overline{M}, \overline{g}) \to (\overline{M}, \overline{g})$ be an isometry and let $\mu_{*p} : T_p\overline{M} \to T_p\overline{M}$ its differential or tangent map. Then μ_{*p} is a linear isometry respect to $\overline{g}_p$, i.e., $\overline{g}(\mu_* v, \mu_* w) = \overline{g}(v, w)$.

Theorem 9 *Let γ be a curve immersed in a Riemannian manifold $(\overline{M}, \overline{g})$ and let $\mu : (\overline{M}, \overline{g}) \to (\overline{M}, \overline{g})$ be an isometry.*

(1) If v is a RM vector field over γ, then $\mu_*(v)$ is a RM vector field over $\mu \circ \gamma$.

(2) If $\{\vec{t}, \vec{v}_1, \ldots \vec{v}_n\}$ is a RM frame over γ, then $\{\mu_*(\vec{t}), \mu_*(\vec{v}_1), \ldots, \mu_*(\vec{v}_n)\}$ is a RM frame over $\mu \circ \gamma$.

Proof. The following claims are well known:

1. If $\vec{t}(t_0)$ is the tangent vector of γ at the point $\gamma(t_0)$, then $\mu_*(\vec{t}(t_0))$ is the tangent vector of $\mu \circ \gamma$ at the point $(\mu \circ \gamma)(t_0)$.
2. If $\vec{v} \in T^{\perp}_{\gamma(t_0)}$, then $\mu_*(\vec{v}) \in T^{\perp}_{(\mu\circ\gamma)(t_0)}$, because μ_* is an isometry.
3. $\mu_*(\overline{\nabla}_X Y) = \overline{\nabla}_{\mu_* X}\ \mu_* Y$ (cfr., e.g [5, p. 161, vol.1]).

Claims (1) and (2) show that μ_* maps tangent (resp. normal) vectors in tangent (resp. normal vectors).

Now, we can easily prove the theorem.

(1) Let v be a RM vector field over γ. Then $D^{\perp}_{\vec{t}} v = 0$, where $\vec{t}$ denotes the tangent vector to γ. We must prove that $D^{\perp}_{\mu_* \vec{t}}\ \mu_* v = 0$.

We have the tangent and normal decomposition:

$$\overline{\nabla}_{\mu_* \vec{t}}\ \mu_* v = -A_{\mu_* v}\ \mu_* \vec{t} + D^{\perp}_{\mu_* \vec{t}}\ \mu_* v$$

and, on the other hand,

$$\overline{\nabla}_{\mu_* \vec{t}}\ \mu_* v = \mu_*(\overline{\nabla}_{\vec{t}}\ v) = \mu_*(-A_v \vec{t} + D^{\perp}_{\vec{t}}\ v) = \mu_*(-A_v \vec{t})$$

which is tangent to $(\mu \circ \gamma)$, thus proving $D^{\perp}_{\mu_* \vec{t}}\ \mu_* v = 0$.

(2) It's a direct consequence of part (1) and claims 1 and 2 at the beginning of the proof.□

Acknowledgments The author wants to express his gratitude to his colleagues Marco Castrillón, Laureano González-Vega, Bert Jütler and Gema Quintana for the useful talks about the theory of RM vectors and frames.

References

[1] Bishop, Richard L. There is more than one way to frame a curve. *Amer. Math. Monthly* **82** (1975), 246–251.

[2] Castrillón López, M.; Fernández Mateos, V.; Muñoz Masqué, J.: Total curvature of curves in Riemannian manifolds. *Differential Geometry and its Applications* 28 (2010) 140–147.

[3] Farouki, Rida T. *Pythagorean-hodograph curves: algebra and geometry inseparable.* Geometry and Computing, 1. Springer, Berlin, 2008.

[4] Gianelli, Carlotta: *Rational Noving Frames on Polynomial Space Curves: Theory and Applications.* PhD Thesis, Università degli studi di Firenze, Florence, 2009.

[5] Kobayashi, Shoshichi; Nomizu, Katsumi: *Foundations of differential geometry.* Vol I and II Interscience Publishers, a division of John Wiley & Sons, New York-London (1963, 1969).

[6] Marí Beffa, G.: Poisson brackets associated to invariant evolutions of Riemannian curves. *Pacific J. Math.* **215** (2004), no. 2, 357–380.

[7] Muñoz Masqué, J.; Rodríguez Sánchez, G. Frenet theorem for spaces of constant curvature. *Geometry from the Pacific Rim* (Singapore, 1994), 253–259, de Gruyter, Berlin, 1997.

[8] M. Özdemir and A. A. Ergin: Parallel Frame of Non-Lightlike Curves. *Missouri J. Math. Sci.*, 20(2)(2008) 1-10. Pekmen

[9] Poor, Walter A. Differential geometric structures. McGraw-Hill Book Co., New York, 1981.